ECE356: Control Theory Midterm Summary

System Formulations

- State Variable Form (non-linear)

$$\begin{cases} \dot{\vec{x}} = f(\vec{x}, u) \\ y = h(\vec{x}, u) \end{cases}$$

$$\dot{x}_1 = f_1(x_1, \dots, x_n, u)$$
$$\dot{x}_2 = f_2(x_1, \dots, x_n, u)$$
$$\vdots$$
$$\dot{x}_n = f_n(x_1, \dots, x_n, u)$$

- LTI State Variable Form

$$\begin{cases} \dot{\vec{x}} = A\vec{x} + \vec{b}u \\ y = \vec{c}^T\vec{x} + du \end{cases}$$

- LTI Input-Output Form:

$$\sum_{i=0}^{n} a_i y^{(i)}(t) = \sum_{j=0}^{m} b_j u^{(j)}(t)$$

$n \geq m$

Linearizing:

① Find equilibrium $(\vec{x}^*, u^*)$ s.t. $f(\vec{x}^*, u^*) = \vec{0}$

② Find Jacobian @ equilibrium: $\frac{\partial f}{\partial x}(x^*) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(x^*) & \cdots & \frac{\partial f_1}{\partial x_n}(x^*) \\ \vdots & & \\ \frac{\partial f_n}{\partial x_1}(x^*) & \cdots & \frac{\partial f_n}{\partial x_n}(x^*) \end{bmatrix}$

③
$$\begin{cases} \dot{\vec{x}} \approx \underbrace{\left[\frac{\partial f}{\partial x}\Big|_{\substack{x=x^* \\ u=u^*}}\right]}_{n\times n} \cdot \delta\vec{x} + \underbrace{\left[\frac{\partial f}{\partial u}\Big|_{\substack{x=x^* \\ u=u^*}}\right]}_{1\times n} \cdot \delta u \\ \delta y \approx \underbrace{\left[\frac{\partial h}{\partial x}(\vec{x}^*, u^*)\right]}_{1\times n} \delta\vec{x} + \underbrace{\left[\frac{\partial h}{\partial u}(x^*, u^*)\right]}_{1\times 1} \delta u \end{cases}$$

↳ $\delta x = x - x^*$

$$\underbrace{A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & & \vdots & & \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix};\ \vec{b} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ 1 \end{bmatrix};\ c = \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_m \\ 0 \\ \vdots \\ 0 \end{bmatrix};\ d = 0}_{\text{one of } \infty \text{ SS realizations of TF.}}$$

$n > m$

Residue Method $\mathcal{L}^{-1}\{\cdot\}$

$$\mathcal{L}^{-1}\{F(s)\} = \sum_{k=1}^{n} \text{Res}(e^{st}F(s), s_k)$$

$$\text{Res}(e^{st}F(s), s_k) = \frac{1}{(n-1)!} \lim_{s \to s_k} \frac{d^{n-1}}{ds^{n-1}}\left[e^{st}F(s) \cdot (s - s_k)^n\right]$$

where:
↳ s_k is $F(s)$'s k^{th} pole location
↳ n is the order of the pole in question.

Transfer Function $G(s)$

- $G(s) = \frac{Y(s)}{U(s)} = \frac{b_m s^m + \dots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \dots + a_1 s + a_0}$ } From I-O Format

⇒ $Y(s) = G(s)U(s)$ ↳ $g(t)$ = "impulse resp" $= \mathcal{L}^{-1}\{G(s)\}$

- $X(s) = (sI_{n\times n} - A)^{-1} B \cdot U(s)$
- $Y(s) = \underbrace{[C(sI_{n\times n} - A)^{-1}B + D]}_{G(s) \text{ from SS}} \cdot U(s)$

e^{At} To Solve $\vec{x}(t)$

$$e^{At} = \mathcal{L}^{-1}\{(sI - A)^{-1}\}$$ ← element-wise $\mathcal{L}\{\cdot\}$

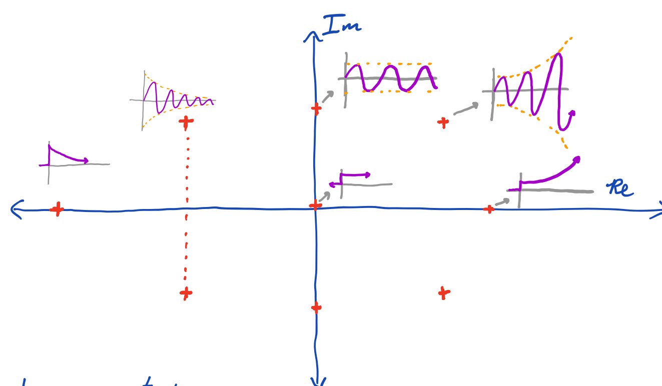
$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)} \cdot Bu(\tau)d\tau$$

$$y(t) = Ce^{At}x(0) + \int_0^t Ce^{A(t-\tau)} \cdot Bu(\tau)d\tau + Du(t)$$

definition: $e^{At} = \sum_{k=0}^{\infty} A^k \frac{t^k}{k!}$

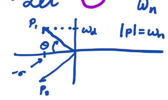
Time Response by Poles

- Let $Y(s) = \frac{N(s)}{D(s)}$ ⇒ Roots of $D(s) \in \mathbb{C}$ are poles of $Y(s)$.
- $y(t)$ = func (poles $Y(s)$)
- Response = lin combo of each pole response:



- $\frac{1}{(s+a)} \to e^{-at}\mathbb{1}(t)$
- $\frac{\omega_n^2}{(s+\sigma)^2 + \omega_d^2} \to y(t) = \frac{b}{\omega_d} e^{-\sigma t}\sin(\omega_d t) \cdot \mathbb{1}(t)$

↳ poles @ $-\sigma \pm j\omega_d$
↳ Let $\zeta = \frac{\sigma}{\omega_n} = \cos\theta$



} N(s) determines weighting of each component.

Eigen e^{At}

- $\text{eig}(A) = V, \Sigma \mid V = \begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix}$

  $\Sigma = \text{diag}(\lambda_1, \dots, \lambda_n)$

- $e^{At} = V\left(\text{diag}(e^{\lambda_i t})\right)V^{-1}$

Modal Decomposition

$$x(t) = \sum_{i=1}^{n} c_i \underbrace{e^{\lambda_i t}}_{\text{eigen mode}} \vec{v}_i$$

scalar const.

where:

$\vec{c} = V^{-1}x(0)$

$\Rightarrow \vec{x}(t)$ is ALWAYS sum of scaled $\vec{v}_i$ vectors

BIBO Stability

- **Def^{n}:** Sys = BIBO stable if:
  $\forall$ bounded $u(t)$, $y(t)$ is bounded $\Big\}$ $\vec{x}(0) = \vec{0}$

$\iff \int_0^\infty |g(t)|\,dt$ is Finite

$\iff$ poles $(G(s)) \in$ OLHP $= \text{Re}\{\text{pole}_i\} < 0$

- Bibo UNStable:
  - ↳ if $u(t)$ shares pole w/ $g(t)$ on j-axis
  - ↳ if pole$_i$ of $G(s) \in$ ORHP ☺
    or $\in j$-axis

Final Value Theorem

$$y(\infty) = \lim_{s \to 0} sF(s)$$

if the limit exists and is finite:

$\iff$

- $Y(s)$ poles only $\in$ OLHP
  + ≤ 1 @ origin

Control Specs (2nd Order)

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$\zeta \in [0,1]$
$\sigma = \zeta\omega_n$
$\omega_d = \omega_n^2 - \sigma^2 = \omega_n^2(1-\zeta^2)$

For $u(t) = \mathbb{1}(t)$: $U(s) = \frac{1}{s}$

- $T_r \simeq \frac{1.8}{\omega_n}$ — time from 10→90% of ss
- $T_p = \frac{\pi}{\omega_d}$ — time till global max
- $T_s \simeq 4/\zeta\omega_n$ — time till $y \in [\pm 2\%\,ss]$ $\forall t > t_s$
- $\%OS = e^{-\zeta\pi/\sqrt{1-\zeta^2}}$ — % overshoot (peak, ss)

Internal Stability

★ only for SS model ☺
"Lyapunov" stability

- resonance: certain osc. in → ∞ output.
- **DEF'N:**
  Sys = stable if $\forall x(0) \in \mathbb{R}^n$
  $\Rightarrow x(t)$ is bounded. → $\exists\, M > 0$ s.t. $|x_i(t)| < M$ $\forall t \geq 0$ $\forall i \in [n]$

  Sys = asymptotically stable if it's stable AND:
  $\forall x(0) \in \mathbb{R}^n$: $\lim_{t \to \infty} x(t) = \vec{0}$

Sys = unstable if $\neg$stable $\iff \exists x(0) \in \mathbb{R}^n$ s.t. $\lim_{t\to\infty} x_i(t) = \infty$
for some i

$\text{Re}\{\lambda_i\} \leq 0 \iff$ Stable — MAYBE, gotta check via modal decomp.

$\underbrace{\text{Re}\{\lambda_i\}}_{\text{eig}(A)} < 0 \iff$ Asymp. Stable

→ depending on $x(0)$, $\text{Re}\{\lambda_i\} = 0$ MIGHT be stable.

⇓

$\Rightarrow$ Note $\{\text{eigvals}(A)\} \supseteq \{\text{poles}(G(s))\}$

so BIBO Instability $\Rightarrow$ Internal Instability

Internal Stability $\Rightarrow$ Bibo Stability

Routh Algorithm

Input: Polyn. $a(s) = s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \dots + a_1 s + a_0 = 0$

Output: T?F: Are all roots $\in$ OLHP?

s^n	a_n	a_{n-2}	a_{n-4}	a_{n-6}	a_{n-8}
s^{n-1}	a_{n-1}	a_{n-3}	a_{n-5}	a_{n-7}	a_{n-9}
s^{n-2}	b_1	b_2	b_3	b_4	$\cdots$
s^{n-3}	c_1	c_2	c_3	c_4	$\cdots$
$\vdots$					

$b_1 = f\begin{bmatrix} a_n & a_{n-2} \\ a_{n-1} & a_{n-3} \end{bmatrix}$

$b_2 = f\begin{bmatrix} a_n & a_{n-4} \\ a_{n-1} & a_{n-5} \end{bmatrix}$

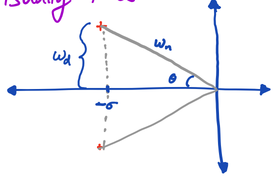
$b_3 = f\begin{bmatrix} a_n & a_{n-6} \\ a_{n-1} & a_{n-5} \end{bmatrix}$

$f\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{-1}{c}(ad - bc)$

Routh Condition:

$\boxed{\#\text{ sign variations in } 1^{st} \text{ column} \equiv \#\text{ poles} \in \text{ORHP}}$

Visually: Poles of $G(s)$:



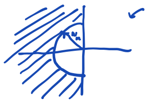
$\zeta = \cos(\theta)$

For $u(t) = \mathbb{1}(t)$:

$\Rightarrow y(t) = \left[1 - e^{-\sigma t}\left(\cos(\omega_d t) + \frac{\sigma}{\omega_d}\sin(\omega_d t)\right)\right] \cdot \mathbb{1}(t)$

Given $T_r \le T_r^d$; $T_s \le T_s^d$; $\%OS \le \%OS^d$:

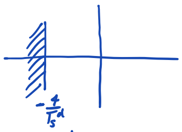
① $\omega_n \ge \frac{1.8}{T_r^d}$ ← "ω_n^d"



← pole locations.

② $\frac{4}{\sigma} = \frac{4}{\omega_n \zeta} < T_s^d$

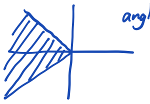
$\Rightarrow \sigma \ge \frac{4}{T_s^d}$ ← "σ^d"



$= -\sigma^d$

③ $\zeta \ge \zeta^d = \frac{-\ln(\%OS^d)}{\sqrt{\pi^2 + \ln(\%OS)^2}}$

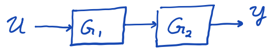
$\cos(\theta) = \zeta$



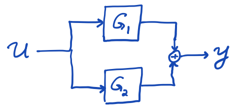
angle of cone $= \cos^{-1}(\zeta)$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

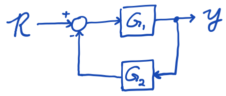
TF INTERCONNECTION



$y = G_1 G_2 u$



$y = (G_1 + G_2)u$



$y = \frac{G_1}{1+G_1G_2} \cdot u$